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by

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ABSTRACT

Several results on Post lattices are presented which generalize theorems found in Epstein [2], and characterizations are given, in the case where the lattice is finite, for two sequences occurring in his definition of Post algebra. These characterizations yield a striking simplification in the computational complexity of an example given in Wojcik [7].

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I. INTRODUCTION

The past several years have seen a number of papers presented dealing with the application of many-valued logics to the theory of digital systems. The underlying algebra employed for the theory was often a finite multivalued Boolean algebra $B(2^m)$ or a Galois field $GF(2^m)$. Rather than these systems, it is the authors' opinion that because of their linear ordering, those distinguished lattices known as chains or n-chains form a more natural mathematical foundation for the theory of n-valued switching devices. The fact that n-chains exist for every positive integer n, whereas finite Boolean algebras of order k exist only for $k = 2^m$ and Galois fields $GF(P^m)$ exist only for powers of primes, seems to indicate that the concept of a chain is more widely applicable to n-valued logic elements than either Boolean algebras or finite fields. The connection between n-chains and finite Post algebras is found in a theorem of Rosenbloom [5], which is stated in Section II. Another important point concerning the relative generality of Boolean algebras, Galois fields and Post algebras is that every Boolean algebra $B(2^m)$ is a Post algebra. This result, which will be established in Section II, shows that finite Post algebras are generalizations of finite Boolean algebras. On the other hand, Galois fields generalize only the trivial switching algebra B(2), since

$B(2^m)$, which contains proper divisors of zero when $m \geq 2$, is not even an integral domain.

Important work has been done by Wojcik [7] and by Wojcik and Metze [8] on n-chains and Post algebras of functions on n-chains for $n \geq 2$. It is the purpose of this report to present several results, primarily concerning finite Post algebras, which greatly simplify some of the work found in Wojcik [7]. In the process, several theorems will be established which generalize results proved by Epstein [2].

II. BASIC THEORY

The following characterization of Post algebras is taken from Epstein [2].

DEFINITION 1: Let n be an integer such that $n \geq 2$. A Post algebra P is a distributive lattice with zero 0 and unit U for which the following conditions hold:

AXIOM 1: There exist n elements $e_o, \ldots, e_{n-1}$ satisfying

(1a) $0 = e_o \leq e_1 \leq \ldots \leq e_{n-1} = U$;

(1b) if $x \in P$ and $x \cdot e_1 = 0$, then $x = 0$;

(1c) if $x \in P$ and $x + e_{i-1} = e_i$ for some i, then $x = e_i$.

AXIOM 2: For each element $x \in P$, there exist n elements $C_o(x), \ldots, C_{n-1}(x)$ in P satisfying

(2a) $C_i(x) \cdot C_j(x) = 0$ if $i \neq j$;

(2b) $\sum_{i=o}^{n-1} C_i(x) = U$;

(2c) $x = \sum_{i=o}^{n-1} (e_i \cdot C_i(x))$.

The two finite sequences of Definition 1 which were alluded to in the Abstract and which we seek to characterize are the elements $e_o, \ldots, e_{n-1}$ and the functions $C_o, \ldots, C_{n-1}$. The first preliminary result we need for this endeavor is a theorem of Rosenbloom [5] which gives the connection between n-chains and finite Post algebras.

THEOREM 2: P is a finite Post algebra if and only if P is isomorphic to a direct product of m n-chains, where $m \geq 1$ and $n \geq 2$ are integers.

For $n \geq 2$, the n-chain will be denoted by P(n) and the direct product of m n-chains by $[P(n)]^m$. Wojcik [7] uses the term Post chain for the n-chain P(n). If x and y are elements of the Post lattice P(n), then $x + y$ is the maximum of x and y and $x \cdot y$ is their minimum.

As we will be working primarily with finite algebras in this report, Theorem 2 will be extemely useful because it gives a representation theory for all finite Post lattices. Thus if the elements of the Post chain P(n) are denoted by $o,1,\ldots,n-1$, where $o < 1 < \ldots < n-1$, then the next result is an immediate consequence of Theorem 2.

COROLLARY 3: If $P \cong [P(n)]^m$, then the elements of P can be represented by those n^m m-tuples of integers, each of whose coordinates lies between o and n-1 inclusive. Also, 0 and U are represented by the m-tuples $\underbrace{o\ldots o}_{m}$ and $\underbrace{(n-1)\ldots(n-1)}_{m}$, respectively.

As a consequence of this corollary, the lattice operations of $+$ and $\cdot$ defined on a finite Post algebra $[P(n)]^m$ are componentwise maximum and minimum functions; that is, maximum and minimum operations on the m Post chains P(n).

Unlike Boolean algebras, not every element of an arbitrary Post algebra P need have a complement. However, it is a well-known result that those elements of P which do have complements form a Boolean algebra, called the underlying Boolean algebra of P and denoted by P_B. In fact it is trivial to verify this. In that P is a distributive lattice, so is P_B. Since P_B is also complemented, it is a Boolean algebra.

We now seek a characterization of the elements of P_B when $P \cong [P(n)]^m$. We begin with the case where $P \cong P(n)$.

LEMMA 4: The Post chain P(n) has B(2) as its underlying Boolean algebra, with o and n-1 being its only complemented elements.

PROOF: The elements $0 = o$ and $U = n-1$ of P(n) satisfy $0 + U = U + 0 = U$ and $0 \cdot U = U \cdot 0 = 0$. Since $0 \leq U$ and both are complemented, they form a Boolean algebra under the operations + and $\cdot$ which is isomorphic to B(2). It will now be shown that o and n-1 are the only elements of P(n) which are complemented. For let $k \in P(n)$ satisfying $o < k < n-1$. If ℓ were the complement of k in P(n), then we would have $k + \ell = \max\{k, \ell\} = U = n-1$ and $k \cdot \ell = \min\{k, \ell\} = 0 = o$. Since $o < k < n-1$, $k + \ell = n-1$ implies $\ell = n-1$ and $k \cdot \ell = o$ implies $\ell = o$. Thus we would have $n-1 = o$; that is, $n = 1$. This would contradict P(n) being a Post chain. Thus o and n-1 are the only complemented elements of P(n) and, hence, $P_B \cong B(2)$ when $P \cong P(n)$.

Armed with this preliminary result, we can now classify those elements of an arbitrary finite Post algebra which are members of its underlying Boolean algebra.

THEOREM 5: The underlying Boolean algebra P_B of the Post algebra $[P(n)]^m$ is $[B(2)]^m$, which is isomorphic to $B(2^m)$. The complemented elements of $[P(n)]^m$ are those m-tuples $x_1 \ldots x_m$, each of whose coordinates is either o or n-1.

PROOF: By Corollary 3, the zero and unit of $[P(n)]^m$ are the m-tuples $0 = \underbrace{o \ldots o}_{m}$ and $U = \underbrace{(n-1) \ldots (n-1)}_{m}$. Let $x = x_1 \ldots x_m$ be an m-tuple, each of whose components is either o or n-1. Define an element $y = y_1 \ldots y_m$ by $y_i = (n-1) - x_i$, $i = 1, \ldots, m$. Then $y_i = o$ if $x_i = n-1$ and $y_i = n-1$ when $x_i = o$. Therefore $(x + y)_i = \max\{x_i, y_i\} = n-1$ and $(x \cdot y)_i = \min\{x_i, y_i\} = o$. Also, $y \in [P(n)]^m$ by Corollary 3. It follows that every m-tuple, each of whose coordinates is either o or n-1, is an element of P_B. We now wish to show that these are the only m-tuples which have complements. To this end, let $z = z_1 \ldots z_m$ be an element of $[P(n)]^m$ having at least one coordinate, say z_i, equal to k, where $o < k < n-1$. In the proof of Theorem 4 it was shown that k has no complement in P(n). Since the operations + and $\cdot$ are performed componentwise on the m-tuples of $[P(n)]^m$, z can have no complement in this algebra. Thus P_B consists of those 2^m m-tuples, each of whose coordinates is o or n-1. By Lemma 4, the underlying Boolean algebra of P(n) consists of o and n-1 and is isomorphic to B(2). Therefore the underlying Boolean algebra of $[P(n)]^m$ is isomorphic to m copies of B(2); that is, if $P \cong [P(n)]^m$ then $P_B \cong [B(2)]^m$. Since there is a unique Boolean algebra of order 2^m for each positive integer m and since the order of $[B(2)]^m$ is 2^m, it follows that $[B(2)]^m \cong B(2^m)$.

This theorem will be extremely useful in our characterization of the functions $C_o, \ldots, C_{n-1}$.

Before proceeding with our work on the classification of the sequences $e_o, \ldots, e_{n-1}$ and $C_o, \ldots, C_{n-1}$, we first establish a result mentioned in Section I concerning the relative generality of Boolean and Post algebras.

THEOREM 6: Every Boolean algebra $B(2^m)$ is a Post algebra.

PROOF: We have seen in the proof of Theorem 5 that $B(2^m) \cong [B(2)]^m$, the direct product of m 2-chains. But by Theorem 2, a product of m n-chains is a Post algebra if $n \geq 2$. Therefore $B(2^m)$ is a Post algebra.

Thus the algebra associated with the n-valued logic of Post is a generalization of the algebra of 2-valued logic. This is not surprising of course, since Post's n-valued logic is a generalization of the standard 2-valued logic.

Returning now to our study of the elements $e_o, \ldots, e_{n-1}$ of a Post algebra P, the next result, which appears in Epstein [2] and whose proof can be found there, will be used for their classification in the case where P is finite.

THEOREM 7: The elements $e_o, \ldots, e_{n-1}$ of a Post algebra are distinct and unique.

The e_i will now be characterized for $[P(n)]^m$ by defining a sequence of elements $d_o, \ldots, d_{n-1}$, showing that this sequence satisfies properties (1a), (1b), (1c) and (2c) of Definition 1, and then using Theorem 7 to show that $d_i = e_i$ for all $i = o, \ldots, n-1$.

<u>THEOREM 8</u>: Let $P \cong [P(n)]^m$. Then the elements $e_o, \ldots, e_{n-1}$ of P are given, for $i = o, \ldots, n-1$, by $e_i = a_{i_1} \ldots a_{i_m}$, where $a_{i_j} = i$ for all $j = 1, \ldots, m$.

<u>PROOF</u>: For $i = o, \ldots, n-1$, let $d_i = a_{i_1} \ldots a_{i_m}$, where $a_{i_j} = i$ for all $j = 1, \ldots, m$. By Corollary 3, each d_i is a member of $[P(n)]^m$. It will be shown that $d_o, \ldots, d_{n-1}$ satisfy properties (1a), (1b), (1c) and (2c) of Definition 1. Obviously the d_i are distinct, and $d_o \leq d_1 \leq \ldots \leq d_{n-1}$ since this relationship holds for each of the m components of each d_i. Also, $d_o = 0$ and $d_{n-1} = U$, the zero and unit elements of $[P(n)]^m$, by Corollary 3. Therefore the elements $d_o, \ldots, d_{n-1}$ satisfy property (1a) of Definition 1.

Now let $x = x_1 \ldots x_m$ be an element of $[P(n)]^m$ and assume that $x \cdot d_1 = (x_1 \ldots x_m) \cdot (d_{1_1} \ldots d_{1_m}) = 0$. Then $(x \cdot d_1)_j = x_j \cdot d_{1_j} = \min \{x_j, 1\} = o$, for all $j = 1, \ldots, m$. Therefore $x_j = o$ for $j = 1, \ldots, m$; that is, $x = 0$. Thus $d_o, \ldots, d_{n-1}$ satisfy (1b) of Definition 1.

Now suppose $x = x_1 \ldots x_m$ is in $[P(n)]^m$ and assume $x + d_{i-1} = (x_1 \ldots x_m) + (d_{(i-1)_1} \ldots d_{(i-1)_m}) = d_i$ for some i, $1 \leq i \leq n-1$. Then $(x + d_{i-1})_j = x_j + d_{(i-1)_j} = \max \{x_j, i-1\} = i$, for all $j = 1, \ldots, m$. It follows that $x_j = i$ for $j = 1, \ldots, m$; that is, $x_j = d_{i_j}$. Therefore $x = d_i$ and, hence, $d_o, \ldots, d_{n-1}$ satisfy (1c) of Definition 1. Axiom (2c), which involves both the elements e_i and the functions C_i, will be established in the proof of Theorem 13. This cannot be done now, as we have not yet introduced the C_i for $[P(n)]^m$. Assuming (2c) for the moment, we have $d_i = e_i$ for all $i = o, \ldots, n-1$; that is, $e_i = a_{i_1} \ldots a_{i_m}$, where $a_{i_j} = i$ for all $j = 1, \ldots, m$.

Thus in $[P(n)]^m$ we have $e_o = \underbrace{o\ldots o}_{m}$, $e_1 = \underbrace{1\ldots 1}_{m}$, ..., $e_{n-1} = \underbrace{(n-1)\ldots(n-1)}_{m}$.

Having characterized the elements e_i, we turn now to the functions $C_o,\ldots,C_{n-1}$. The first result we will need for their characterization is a theorem from Epstein [2] giving a necessary and sufficient condition for an element of a Post algebra P to be a member of its underlying Boolean algebra P_B.

<u>THEOREM 9</u>: If $x \in P$, then $x \in P_B$ if and only if $x = C_i(y)$ for some i, $o \leq i \leq n-1$, and some element y of P.

Part of what this theorem tells us is that every function C_i defined on a Post algebra P maps P into P_B, its underlying Boolean algebra. Once we have characterized the functions $C_o,\ldots,C_{n-1}$ defined on a Post algebra $[P(n)]^m$, it will be possible to show that each such function C_i maps P onto P_B; that is, the range of each C_i is the entire underlying Boolean algebra of P. To facilitate the characterization of the sequence of functions $C_o,\ldots,C_{n-1}$, one additional result, taken from Epstein [2], will be used.

<u>THEOREM 10</u>: If $b \in P_B$ and $b\cdot e_i = b\cdot e_j$ for some i and j with $i < j$, then $b = 0$.

Using Theorems 9 and 10, the following result, which is a generalization of a theorem given in Epstein [2], can now be established. Epstein's result will then follow as a corollary.

<u>THEOREM 11</u>: For each Post algebra P, the sequence of functions $C_o,\ldots,C_{n-1}$ defined on P is unique.

<u>PROOF</u>: The plan of the proof is to let $D_o,\ldots,D_{n-1}$ be another such sequence of functions, and then to show that D_i must be the function C_i for all $i = o,\ldots,n-1$. Begin by letting $D_o,\ldots,D_{n-1}$ be a sequence of functions which satisfy properties (2a), (2b) and (2c) of Definition 1. It will be shown that $C_i(x) = D_i(x)$ for all $i = o,\ldots,n-1$ and for all $x \in P$. To this end, let $x \in P$ and consider the sequences of function values $C_o(x),\ldots,C_{n-1}(x)$ and $D_o(x),\ldots,D_{n-1}(x)$. By (2c) of Definition 1, $\sum_{k=o}^{n-1}(e_k\cdot C_k(x)) = \sum_{k=o}^{n-1}(e_k\cdot D_k(x))$. Therefore, if $i \neq j$,

$C_i(x)\cdot D_j(x)\cdot \sum_{k=o}^{n-1}(e_k\cdot C_k(x)) = C_i(x)\cdot D_j(x)\cdot \sum_{k=o}^{n-1}(e_k\cdot D_k(x))$. But by (2a) of Definition 1, $C_i(x)\cdot D_j(x)\cdot \sum_{k=o}^{n-1}(e_k\cdot C_k(x)) = e_i\cdot C_i(x)\cdot D_j(x)$ and

$C_i(x)\cdot D_j(x)\cdot \sum_{k=o}^{n-1}(e_k\cdot D_k(x)) = e_j\cdot C_i(x)\cdot D_j(x)$. Thus $e_i\cdot C_i(x)\cdot D_j(x)$

$= e_j\cdot C_i(x)\cdot D_j(x)$. By Theorem 9, $C_i(x)$ and $D_j(x)$ are members of P_B. Hence, $C_i(x)\cdot D_j(x) \in P_B$. Then by Theorem 10, $C_i(x)\cdot D_j(x) = 0$. From property (2a) of Definition 1 we have that $\sum_{k=o}^{n-1} C_k(x) = U$. Thus for every j, $o \leq j \leq n-1$, $D_j(x) = D_j(x)\cdot \sum_{k=o}^{n-1} C_k(x) = D_j(x)\cdot C_j(x)$, since $C_i(x)\cdot D_j(x)$

$= 0$ for $i \neq j$. Similarly, $\sum_{k=o}^{n-1} D_k(x) = U$ implies that

$C_j(x) = C_j(x) \cdot \sum_{k=o}^{n-1} D_k(x) = C_j(x) \cdot D_j(x)$. Consequently, $C_j(x) = D_j(x)$ for all $j = o,\ldots,n-1$. Since $\bar{x}$ was chosen arbitrarily from P, it follows that $C_j = D_j$ for all $j = o,\ldots,n-1$. Thus the sequence of functions $C_o,\ldots,C_{n-1}$ defined on P is unique.

Epstein's result on the uniqueness of the elements $C_o(x),\ldots, C_{n-1}(x)$ is now obvious.

<u>COROLLARY 12</u>: For each $x \in P$, the sequence of elements $C_o(x),\ldots,C_{n-1}(x)$ is unique.

<u>PROOF</u>: Since the sequence of functions $C_o,\ldots,C_{n-1}$ is unique, by Theorem 11, and since each C_i, $i = o,\ldots,n-1$, is a function; that is, each C_i is well-defined, it follows that the sequence of function values $C_o(x),\ldots, C_{n-1}(x)$ is unique for a particular x in P.

Notice that although the $C_i(x)$ have been shown to be unique for each choice of the element $\bar{x}$ in P, they are not alleged to be unique for all x in P. In fact they are not; as the variable x changes from one element of P to another, the function values $C_i(x)$ may also vary from one element of P_B to another. This will be seen clearly in the example given in Section III once we have characterized the functions $C_o,\ldots,C_{n-1}$ for $[P(n)]^m$. Unlike the elements $C_o(x),\ldots,C_{n-1}(x)$, the functions $C_o,\ldots,C_{n-1}$ are unique for the entire Post algebra P and are in no way dependent on the choice of the argument x.

We are now ready to give a useful characterization of the functions $C_o,\ldots,C_{n-1}$ defined on a finite Post algebra.

THEOREM 13: Let $P \cong [P(n)]^m$. Then each C_i of the sequence of functions $C_o, \ldots, C_{n-1}$ can be defined for $x = x_1 \ldots x_m$ in P by $C_i(x) = C_i(x_1 \ldots x_m)$ $= y_{i_1} \ldots y_{i_m}$, where for each k, $1 \leq k \leq m$,

$$y_{i_k} = \begin{cases} n-1 \text{ if } x_k = i \\ o \text{ if } x_k \neq i. \end{cases}$$

PROOF: Let the sequence of functions $D_o, \ldots, D_{n-1}$, where D_i maps P into P for all $i = o, \ldots, n-1$, be defined for $x = x_1 \ldots x_m$ in P by $D_i(x)$ $= D_i(x_1 \ldots x_m) = y_{i_1} \ldots y_{i_m}$, where for each k, $1 \leq k \leq m$,

$$y_{i_k} = \begin{cases} n-1 \text{ if } x_k = i \\ o \text{ if } x_k \neq i. \end{cases}$$

It will be shown that the $D_i(x)$ satisfy properties (2a), (2b), and (2c) of Definition 1. We first establish (2a). If $i \neq j$, $D_i(x) \cdot D_j(x)$ $= D_i(x_1 \ldots x_m) \cdot D_j(x_1 \ldots x_m) = (y_{i_1} \ldots y_{i_m}) \cdot (y_{j_1} \ldots y_{j_m})$. We want to show that $D_i(x) \cdot D_j(x) = 0$. Now $(D_i(x) \cdot D_j(x))_k = y_{i_k} \cdot y_{j_k} = \min \{y_{i_k}, y_{j_k}\}$. If $y_{i_k} = o$, then $(D_i(x) \cdot D_j(x))_k = o$. On the other hand, if $y_{i_k} \neq o$, then it must be that $y_{i_k} = n-1$. Hence, $x_k = i$ and, since $i \neq j$, $x_k \neq j$. Therefore $y_{j_k} = o$. So again, $(D_i(x) \cdot D_j(x))_k = \min \{y_{i_k}, y_{j_k}\} = o$. Thus in either case, $(D_i(x) \cdot D_j(x))_k = o$. Since this holds for all $k = 1, \ldots, m$, it follows that $D_i(x) \cdot D_j(x) = \underbrace{o \ldots o}_{m} = 0$. Therefore the elements $D_i(x)$ satisfy property (2a) of Definition 1.

For (2b) we must show that $\sum_{i=o}^{n-1} D_i(x) = U$. By definition, $\sum_{i=o}^{n-1} D_i(x)$ $= D_o(x) +\ldots+ D_{n-1}(x) = (y_{o_1} \ldots y_{o_m}) +\ldots+ (y_{(n-1)_1} \ldots y_{(n-1)_m})$. Thus $\left(\sum_{i=o}^{n-1} D_i(x)\right)_k = y_{o_k} +\ldots+ y_{(n-1)_k}$. By Corollary 3, every coordinate of x may be assumed to have a value between o and n-1 inclusive. Let $x_k = \ell$, where $o \leq \ell \leq n-1$. Then $y_{\ell_k} = n-1$ and, incidentally, $y_{j_k} = o$ for all $j = 1, \ldots, \ell - 1, \ell + 1, \ldots, m$. Therefore $\left(\sum_{i=o}^{n-1} D_i(x)\right)_k = y_{o_k} +\ldots+ y_{(\ell-1)_k}$ $+ y_{\ell_k} + y_{(\ell+1)_k} +\ldots+ y_{(n-1)_k} = o + \ldots + o + (n-1) + o + \ldots + o = n-1$. Since this holds for all $k = 1, \ldots, m$, it follows that $\sum_{i=o}^{n-1} D_i(x)$ $= \underbrace{(n-1) \ldots (n-1)}_{m} = U$. Thus the $D_i(x)$ satisfy (2b) of Definition 1.

For (2c) we must show that $\sum_{i=o}^{n-1} (d_i \cdot C_i(x)) = x$. Using the definition of the elements $d_o, \ldots, d_{n-1}$ for $[P(n)]^m$, we have $\sum_{i=o}^{n-1}(d_i \cdot D_i(x)) = d_o \cdot D_o(x) +\ldots+ d_{n-1} \cdot D_{n-1}(x) = \underbrace{(o \ldots o)}_{m} \cdot (y_{o_1} \ldots y_{o_m}) +\ldots+$ $(\underbrace{(n-1) \ldots (n-1)}_{m}) \cdot (y_{(n-1)_1} \ldots y_{(n-1)_m})$. Thus $\left(\sum_{i=o}^{n-1} (d_i \cdot D_i(x))\right)_k$ $= o \cdot y_{o_k} +\ldots+ (n-1) \cdot y_{(n-1)_k}$. Using Corollary 3 as was done for (2b), we may assume $x_k = \ell$, where $o \leq \ell \leq n-1$. Then $y_{\ell_k} = n-1$ and, incidentally, $y_{j_k} = o$ for all $j = 1, \ldots, \ell-1, \ell+1, \ldots, m$. Hence, $\left(\sum_{i=o}^{n-1} (d_i \cdot D_i(x))\right)_k$ $= o \cdot y_{o_k} +\ldots+ (\ell-1) \cdot y_{(\ell-1)_k} + \ell \cdot y_{\ell_k} + (\ell+1) \cdot y_{(\ell+1)_k} +\ldots+ (n-1) \cdot y_{(n-1)_k}$ $= o \cdot o +\ldots+ (\ell-1) \cdot o + \ell \cdot (n-1) + (\ell+1) \cdot o +\ldots+ (n-1) \cdot o = \ell = x_k$. Since this holds for all $k = 1, \ldots, m$, it follows that $\sum_{i=o}^{n-1} (d_i \cdot D_i(x))$

$= x_1 \ldots x_k \ldots x_m = x$. Therefore the elements $d_o, \ldots, d_{n-1}$ and $D_o(x), \ldots,$ $D_{n-1}(x)$ satisfy (2c) of Definition 1. This completes the proof of Theorem 8.

We have now shown that $D_o(x), \ldots, D_{n-1}(x)$ satisfy properties (2a), (2b) and (2c) of Definition 1. By Corollary 12, the sequence $C_o(x), \ldots,$ $C_{n-1}(x)$ satisfying these three properties is unique. So it must be the case that $C_o(x) = D_o(x), \ldots, C_{n-1}(x) = D_{n-1}(x)$. Since this is true for all x in P, it follows that $C_o = D_o, \ldots, C_{n-1} = D_{n-1}$.

It is Theorems 8 and 13 that will be used to help us simplify some of the calculations carried out in Wojcik [7]. Before presenting this simplification, however, we will first prove the strengthened part of Theorem 9 which shows that each of the functions $C_o, \ldots, C_{n-1}$ defined on $[P(n)]^m$ maps this lattice onto its underlying Boolean algebra.

THEOREM 14: Let $P \cong [P(n)]^m$. Each of the functions $C_o, \ldots, C_{n-1}$ defined on P maps it onto its underlying Boolean algebra; that is, the range of each C_i is all of P_B.

PROOF: Let $b \in P_B$. By Theorem 5, b is an m-tuple whose components are o's and (n-1)'s. Let C_i be one of the functions $C_o, \ldots, C_{n-1}$. It must be shown that there exists an element x in P such that $C_i(x) = b$. If $b = b_1 \ldots b_m$, we claim the required x is given by the m-tuple $x_1 \ldots x_m$ where, for each $j = 1, \ldots, m$, $x_j = i$ if $b_j = n-1$ and $x_j = \ell$, for some ℓ satisfying $o \leq \ell \leq n-1$ and $\ell \neq i$, if $b_j = o$. Then $x \in [P(n)]^m$ by Corollary 3, and $C_i(x) = C_i(x_1 \ldots x_m) = b_1 \ldots b_m = b$ by Theorem 13.

Thus $x = x_1 \ldots x_m$ satisfies $C_i(x) = b$. Since b was chosen arbitrarily from P_B, it follows that each function C_i maps P onto P_B.

III. THE EXAMPLE $[P(3)]^3$

The lattice $[P(3)]^3 = P(3) \times P(3) \times P(3)$ is a Post algebra by Theorem 2. This example is considered at length in Wojcik [7] and it is some of his work with this lattice that we propose to simplify. Using the theory that has been developed in Section II, we can gain a great deal of information about $[P(3)]^3$. According to Corollary 3, every element of this algebra can be represented by a 3-tuple of integers, each of whose coordinates is either 0,1, or 2. Corollary 3 also tells us that the zero element 0 and the unit element U are the 3-tuples 000 and 222 respectively. By Theorem 5, the underlying Boolean algebra of $[P(3)]^3$ is isomorphic to $B(2^3)$ and consists of the elements 000, 002, 020, 022, 200, 202, 220 and 222. Using Theorem 8 we obtain $e_o = 000$, $e_1 = 111$ and $e_2 = 222$. The Hasse diagram for $[P(3)]^3$ is given in Figure 1 and the diagram of its underlying Boolean algebra in Figure 2. These diagrams were obtained by using the relationship $x \leq y \Leftrightarrow x \cdot y = x \Leftrightarrow x + y = y$, which is true of every lattice. Since the operations + and $\cdot$ of $[P(n)]^m$ are componentwise maximum and minimum functions by Corollary 3, the inequality $\leq$ is also defined componentwise on the m-tuples of this lattice.

Let us now calculate $C_o(x)$, $C_1(x)$ and $C_2(x)$ for an element x of $[P(3)]^3$. If we select the element $x = 121$ from this algebra, these three function values can be determined in the following way, which is essentially that used in Wojcik [7] and in Wojcik and Metze [8]:

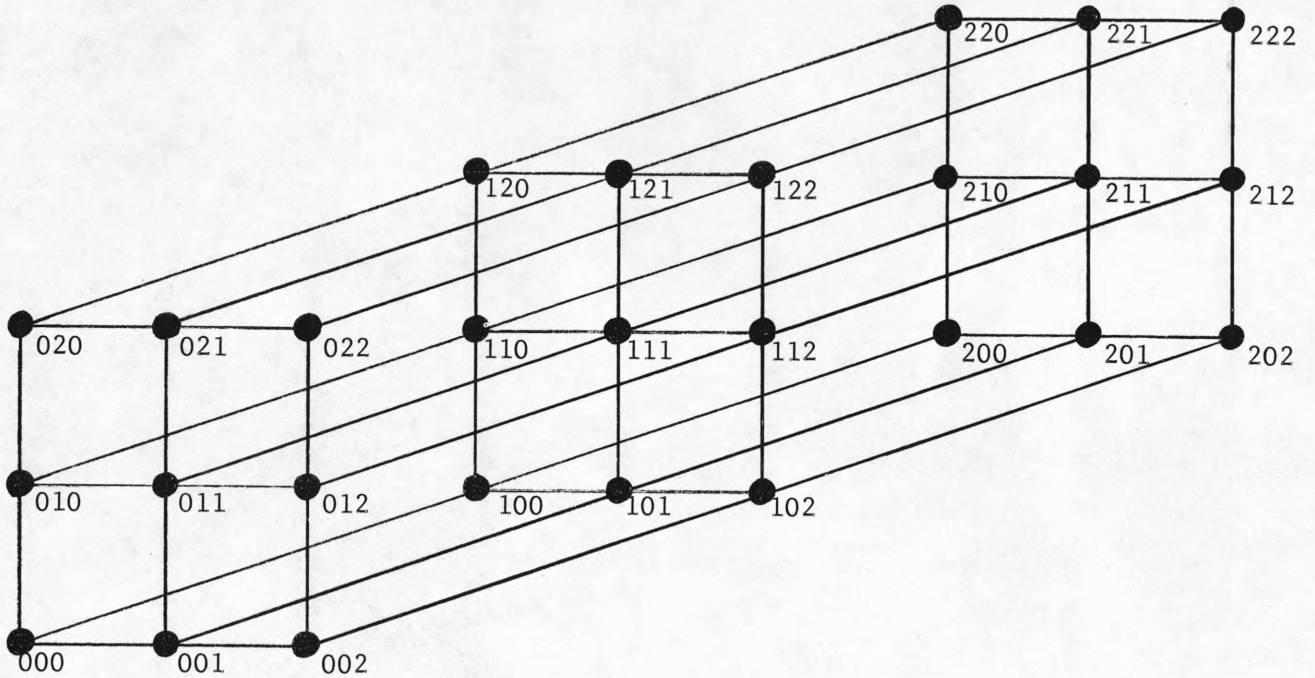

Figure 1. The Post Algebra $[P(3)]^3$

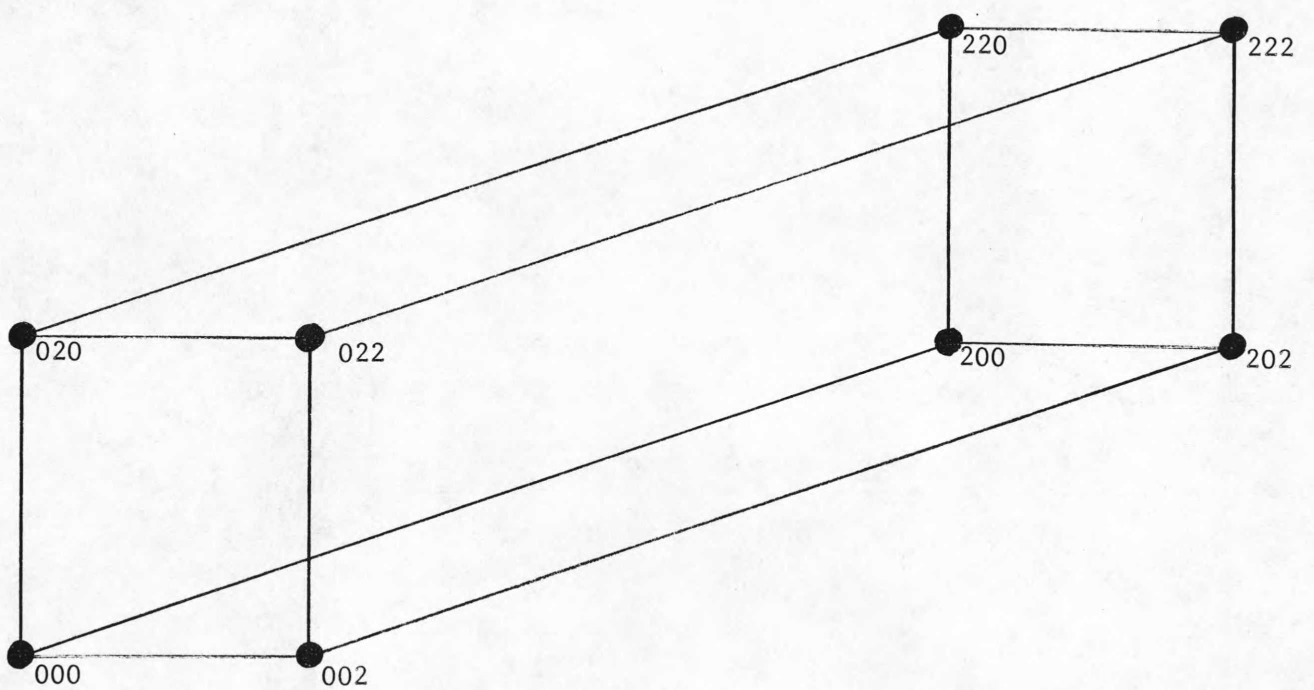

Figure 2. The Underlying Boolean Algebra of $[P(3)]^3$

By (2c) of Definition 1 we have, for an arbitrary element $x = x_1x_2x_3$ of $[P(3)]^3$,

$$\text{(1)} \quad x_1x_2x_3 = e_o \cdot C_o(x_1x_2x_3) + e_1 \cdot C_1(x_1x_2x_3) + e_2 \cdot C_2(x_1x_2x_3)$$
$$= 111 \cdot C_1(x_1x_2x_3) + 222 \cdot C_2(x_1x_2x_3).$$

From (2b) of Definition 1 it follows that

$$\text{(2)} \quad C_o(x_1x_2x_3) + C_1(x_1x_2x_3) + C_2(x_1x_2x_3) = 222.$$

Also, by (2a) of Definition 1 we have the relations

$$\text{(3)} \quad C_o(x_1x_2x_3) \cdot C_1(x_1x_2x_3) = 000,$$
$$C_o(x_1x_2x_3) \cdot C_2(x_1x_2x_3) = 000,$$
$$C_1(x_1x_2x_3) \cdot C_2(x_1x_2x_3) = 000.$$

Now let us adopt the following notation for the function values we are seeking:

$$C_o(x_1x_2x_3) = a_1a_2a_3,$$
$$C_1(x_1x_2x_3) = a_4a_5a_6,$$
$$C_2(x_1x_2x_3) = a_7a_8a_9,$$

where a_j, for all $j = 1,\ldots, 9$, is a member of the underlying Boolean algebra of $P(3)$, which consists of the elements 0 and 2. This follows because the value $C_i(x_1x_2x_3)$, for $i = 0, 1, 2$, is a member of the underlying Boolean algebra of $[P(3)]^3$. From (1), (2) and (3) we have the following relations:

(4) $111 \cdot a_4a_5a_6 + 222 \cdot a_7a_8a_9 = x_1x_2x_3$,

(5) $a_1a_2a_3 + a_4a_5a_6 + a_7a_8a_9 = 222$,

(6) $a_1a_2a_3 \cdot a_4a_5a_6 = 000$,

(7) $a_1a_2a_3 \cdot a_7a_8a_9 = 000$,

(8) $a_4a_5a_6 \cdot a_7a_8a_9 = 000$.

Note that these relations must also hold componentwise. Thus if we now let $x = 121$, from equation (4) we derive the following:

(a) $1 \cdot a_4 + 2 \cdot a_7 = 1$.

This implies that $a_7 \neq 2$ and, hence, $\underline{a_7 = 0}$ because each a_j, $1 \leq j \leq 9$, is a member of the underlying Boolean algebra of $P(3)$; that is, each a_j is either 0 or 2. With $a_7 = 0$ and with a_4 necessarily being 0 or 2, we have $\underline{a_4 = 2}$.

(b) $1 \cdot a_5 + 2 \cdot a_8 = 2$.

This implies $\underline{a_8 = 2}$ because $1 \cdot a_5$ can never be 2; but it gives no information about a_5.

(c) $1 \cdot a_6 + 2 \cdot a_9 = 1.$

This implies $a_9 \neq 2$. Since a_9 must be either 0 or 2, we have $\underline{a_9 = 0}$. This in turn implies that $\underline{a_6 = 2}$ since either $a_6 = 0$ or $a_6 = 2$.

From relation (6) we have the following:

(a) $a_1 \cdot a_4 = 0.$

Since $a_4 = 2$, this implies $\underline{a_1 = 0}$.

(b) $a_2 \cdot a_5 = 0.$

This yields no new information.

(c) $a_3 \cdot a_6 = 0.$

Since $a_6 = 2$, this implies $\underline{a_3 = 0}$.

From the second components of relations (7) and (8) we obtain, respectively,

(a) $a_2 \cdot a_8 = 0.$

Since $a_8 = 2$, this yields $\underline{a_2 = 0}$.

(b) $a_5 \cdot a_8 = 0.$

Since $a_8 = 2$, we must have $\underline{a_5 = 0}$.

Thus we have the function values

$$C_o(121) = a_1a_2a_3 = 000,$$

$$C_1(121) = a_4a_5a_6 = 202,$$

$$C_2(121) = a_7a_8a_9 = 020.$$

This determination of the values of the a_j, $j = 1,\ldots,9$, while correct, is long and tedious. With the availability of Theorem 13, the determination of the elements $C_i(x)$ is immediate: $C_o(121) = 000$, $C_1(121) = 202$, and $C_2(121) = 020$. Table 1 contains the values $C_o(x)$, $C_1(x)$ and $C_2(x)$ for every element $x = x_1x_2x_3$ of $[P(3)]^3$. Note that the $C_i(x)$, $i = 0,1,2$, satisfy (2a), (2b) and (2c) of Definition 1. The remark we made following Theorem 12 concerning the fact that the $C_i(x)$ are not unique for all x in P can also be clearly seen in Table 1 by the changing of entries in rows of the table as the value of x changes. On the other hand, the functions C_o, C_1 and C_2 are unique for the entire Post algebra $[P(3)]^3$. Table 1 also illustrates Theorem 14, showing that every element of the underlying Boolean algebra $B(2^3)$ of $[P(3)]^3$ is in the range of each function C_i, $i = 0,1,2$.

In conclusion, we remark that the simplification we have given for $[P(3)]^3$ is perfectly general for every lattice $[P(n)]^m$ because the theorems we have proved hold for every finite Post algebra. The amount of work saved by applying Theorem 13 to $[P(3)]^3$ gives only a small indication of the labor which would be saved in a computation of the $C_i(x)$ for an algebra $[P(n)]^m$ where n and m are relatively large.

$x_1x_2x_3$	$C_o(x_1x_2x_3)$	$C_1(x_1x_2x_3)$	$C_2(x_1x_2x_3)$
000	222	000	000
001	220	002	000
002	220	000	002
010	202	020	000
011	200	022	000
012	200	020	002
020	202	000	020
021	200	002	020
022	200	000	022
100	022	200	000
101	020	202	000
102	020	200	002
110	002	220	000
111	000	222	000
112	000	220	002
120	002	200	020
121	000	202	020
122	000	200	022
200	022	000	200
201	020	002	200
202	020	000	202
210	002	020	200
211	000	022	200
212	000	020	202
220	002	000	220
221	000	002	220
222	000	000	222

<u>TABLE 1</u>

The Functions C_o, C_1, C_2 for $[P(3)]^3$

REFERENCES

1. Birkhoff, G., Lattice Theory, American Mathematical Society Colloquium Publication 25, revised edition, New York, 1948.

2. Epstein, G., "The Lattice Theory of Post Algebras," Transactions of the American Mathematical Society, Vol. 95, pp. 300-317, 1960.

3. Metze, G. A., "An Application of Multivalued Logic Systems to Circuits," Proceedings of a Symposium on Circuit Analysis, University of Illinois, Urbana, pp. 11-1 to 11-14, 1955.

4. Post, E., "Introduction to a General Theory of Elementary Propositions," American Journal of Mathematics, Vol. 43, pp. 163-185, 1921.

5. Rosenbloom, P. C., "Post Algebras. I. Postulates and General Theory," American Journal of Mathematics, Vol. 64, pp. 167-188, 1942.

6. Traczyk, T., "Axioms and Some Properties of Post Algebras," Colloquium Mathematicum, Vol. 10, pp. 193-209, 1963.

7. Wojcik, A. S., "Relationships Between Post and Boolean Algebras with Application to Multivalued Switching Theory," Coordinated Science Laboratory Report R-512, University of Illinois, Urbana, 1971.

8. Wojcik, A. S., and Metze, G. A., "An Analysis of Some Relationships Between Post and Boolean Algebras with Application to the Minimization of Higher-Order Boolean Functions," to be published.

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13. ABSTRACT

Several results on Post lattices are presented which generalize theorems found in Epstein [2], and characterizations are given, in the case where the lattice is finite, for two sequences occurring in his definition of Post algebra. These characteristics yield a striking simplification in the computational complexity of an example given in Wojcik [7].

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